# On geometric structure of phase portraits of some piecewise linear dynamical systems*


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**Abstract**

We construct an invariant surface in a non-invariant domain of phase portrait of one piecewise linear dynamical system which appears naturally in gene networks modeling. This surface does not intersect the invariant neighborhood of a cycle which we have found in that phase portrait earlier, neither it contains other cycles of this system. All trajectories of this system contained in this surface are attracted to a point which plays role of an equilibrium point for smooth dynamical systems of this type.



*2000 Mathematics Subject Classification.* **37B35**. 37N25.
*Keywords.* nonlinear dynamical systems, invariant surfaces, phase portraits, gene networks models, Frobenius-Perron theorem, homotopy groups of spheres.


## Introduction

We continue our studies of nonlinear dynamical systems which simulate functioning of some simple circular gene networks. A typical example of these systems has the form

$$\frac{dm_1}{dt} = f_1(p_3) - k_1 m_1; \ \ \frac{dp_1}{dt} = g_1(m_1) - k_2 p_1; \ \ \frac{dm_2}{dt} = f_2(p_1) - k_3 m_2;$$

$$\frac{dp_2}{dt} = g_2(m_2) - k_4 p_2; \ \ \frac{dm_3}{dt} = f_3(p_2) - k_5 m_3; \ \ \frac{dp_3}{dt} = g_3(m_3) - k_6 p_3. \tag{1}$$

Here all function $f_j$, $g_j$ are assumed to be monotonic and non-negative, parameters $k_j$ are positive, non-negative variables $p_1$, $p_2$, $p_3$, and $m_1$, $m_2$, $m_3$ denote concentrations of three proteins, respectively, of three mRNAs in this circular gene network. The functions $f_1$, $f_2$, $f_3$ are decreasing, this means that they describe negative feedbacks. The functions $g_1$, $g_2$, $g_3$ are increasing, they correspond to positive feedbacks in the gene network.

In one very particular case

$$f_i(w) = \alpha \cdot (1 + w^n)^{-1} + \alpha_0; \ \ g_i(u) = \mu u; \ \ k_{2i-1} = 1; \ \ k_{2i} = \mu > 0; \ \ i = 1, 2, 3, \tag{2}$$

the system (1) was introduced in [1, 2], and later studied in [3, 4]. So, the system (1), (2) is symmetric with respect to cyclic permutations of the pairs

$$(m_1, p_1) \to (m_2, p_2) \to (m_3, p_3) \to (m_1, p_1).$$

*In memory of Roin Nadiradze


**Tbilisi Mathematical Journal** Special Issue (7 - 2021), pp. 49–56.
Tbilisi Centre for Mathematical Sciences.

*Received by the editors:* 11 December 2019.
*Accepted for publication:* 03 April 2020.

Some asymmetric higher-dimensional analogues of the system (1) were considered in [5, 6, 7]. The main result of the present paper consists of construction of an invariant surface in a non-invariant domain of the phase portrait of one dynamical system similar to (1). These studies are motivated by the fact that such invariant surfaces are usually contained in boundaries of attraction basins of stable cycles and other attractors in these phase portraits. In particular, such a surface appears in a model of one natural gene network, see [8], and this observation follows the example composed by S.Smale [9].

## 1 PL dynamical systems

We consider now piecewise linear version of the system (1):

$$\frac{dx_0}{dt} = L_0(x_5) - k_0x_0; \quad \frac{dx_1}{dt} = \Gamma_1(x_0) - k_1x_1; \quad \frac{dx_2}{dt} = L_2(x_1) - k_2x_2;$$

$$\frac{dx_3}{dt} = \Gamma_3(x_2) - k_3x_3; \quad \frac{dx_4}{dt} = L_4(x_3) - k_4x_4; \quad \frac{dx_5}{dt} = \Gamma_5(x_4) - k_5x_5, \tag{3}$$

where monotonic step functions $L_{2i}$ and $\Gamma_{2i+1}$ in the equations are defined as follows:

$$L_{2i}(w) = k_{2i}(a_{2i} - 1), \text{ for } -1 \le w \le 0; \quad L_{2i}(w) = -1 \text{ for } w > 0;$$

$$\Gamma_{2i+1}(u) = k_{2i+1}(a_{2i+1} - 1), \text{ for } u > 0; \quad \Gamma_{2i+1}(u) = -1 \text{ for } -1 \le u \le 0.$$

All the variables satisfy the conditions $x_j + 1 \ge 0$, all parameters are positive, $a_j > 1$. Here and below $i = 0, 1, 2;\ j = 0, 1, \ldots, 5$.

Analogous gene networks models with step functions in right hand sides of the equations were studied in [10, 11] and used Boolean analysis combined with qualitative theory of ODE, see also [12, 13, 14] and references therein.

The main aim of the publications cited above is description of phase portraits of these dynamical systems in order to detect their closed trajectories (cycles) and to localize them there. Important problems of construction of integral submanifolds, detection of attractors, the basins of their attraction, and other geometric characteristics of these phase portraits appear here naturally. We study in this paper one case of the first of these problems.

Consider the parallelepiped $Q^6 = \prod_{j=0}^{j=5}[-1, a_j - 1] \subset \mathbb{R}^6$. The origin of $\mathbb{R}^6$ is contained in $Q^6$, thus, the coordinate hyperplanes $x_j = 0$ subdivide $Q^6$ to 64 smaller parallelepipeds which we call *blocks* for brevity, and enumerate by binary multi-indices $\{\varepsilon_0\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4\varepsilon_5\}$ as follows:

$$\varepsilon_j = 0, \text{ if } x_j \le 0 \text{ for all points of this block}; \quad \varepsilon_j = 1 \text{ otherwise}. \tag{4}$$

Similar discretizations of phase portraits for other gene networks models (smooth, piecewise linear, odd- and even-dimensional) were considered in [7, 15, 16]. As it was shown there, Lemmas 1.1 and 1.2 follow immediately from calculation of derivatives $\dot{x}_j$ on the faces of $Q^6$ and on the coordinate planes $x_j = 0$.

**Lemma 1.1.** *$Q^6$ is positively invariant domain of the system* (3).

**Lemma 1.2.** *For any pair $B_1$, $B_2$ of adjacent blocks of this decomposition, trajectories of all points of their common 5-dimensional face $F = B_1 \cap B_2$ pass either from $B_1$ to $B_2$ or from $B_2$ to $B_1$.*

We denote below these transitions of trajectories as $B_1 \to B_2$, respectively, $B_2 \to B_1$. It follows from Lemma 1.2 that the graph $G$ composed by edges of boolean 6-dimensional cube $\mathcal{B}^6$ with the vertices $\{\varepsilon_0\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4\varepsilon_5\}$ can be oriented according to directions of these arrows. Here $\varepsilon_j = 0$ or $\varepsilon_j = 1$, as in (4).

**Definition.** *The valence $V(B)$ of a block $B$ is a number of its 5-dimensional faces such that trajectories of their points come out of $B$ to its adjacent blocks.*

In terms of graph theory this means that the vertex of the cube $\mathcal{B}^6$ corresponding to the block $B$ has outgoing degree $V(B)$.

**Remark 1.** *This definition is naturally formulated for dynamical systems of different dimensions and "smoothness" analogous to* (1), (3). *The cycles of these systems described in* [6, 7, 17] *are contained in the unions of* 1*-valent blocks and travel from block to blocks according to arrows of some circular diagrams. For one analogous gene network model considered in* [10], *such a diagram is called State Transition Diagram. It was shown in* [11] *that the systems* (1) *and* (3) *have cycles in the domain $Q^6$, and these cycles follow arrows of the diagram*

$$\begin{tikzcd}
\{110011\} \arrow[r] & \{010011\} \arrow[r] & \{000011\} \arrow[r] & \{001011\} \arrow[d] \\
\{110010\} \arrow[u] & & & \{001111\} \arrow[d] \\
\{110000\} \arrow[u] & & & \{001101\} \arrow[d] \\
\{110100\} \arrow[u] & \{111100\} \arrow[l] & \{101100\} \arrow[l] & \{001100\} \arrow[l]
\end{tikzcd} \tag{5}$$

All blocks listed here have valence 1. Let $W_1$ be the union of these 1-valent blocks. This is an invariant domain of the systems (1) and (3), cf. [6].

It follows from [18] that for the 6-dimensional systems (1) and (3) valence of blocks of partition of $Q^6$ equals either 1, or 3, or 5, and it is not difficult to verify that all 1-valent blocks are listed in the diagram (5), and the diagram (6) below contains all 5-valent blocks. Let $W_3$ and $W_5$ be the unions of 3-valent, respectively, 5-valent blocks. The non-invariant domain $W_3$ contains 40 blocks.

## 2 Non-invariant domain $W_5$

The main goal of the present paper is description of behavior of trajectories of the system (3) in the non-invariant domain $W_5 \subset Q^6 \setminus (W_1 \cup W_3)$. The domain $W_5$ consists of 12 blocks which are connected by the cyclic diagram

$$\begin{array}{ccccccc}
\{111001\} & \longrightarrow & \{011001\} & \longrightarrow & \{011000\} & \longrightarrow & \{011010\} \\
\uparrow & & & & & & \downarrow \\
\{101001\} & & & & & & \{011110\} \\
\uparrow & & & & & & \downarrow \\
\{100001\} & & & & & & \{010110\} \\
\uparrow & & & & & & \downarrow \\
\{100101\} & \longleftarrow & \{100111\} & \longleftarrow & \{100110\} & \longleftarrow & \{000110\}
\end{array} \tag{6}$$

Each arrow of this diagram shows a possible direction of transition of trajectories from block to block. In contrast with the diagram (5) where each arrow shows the unique direction, most of the transitions of trajectories of points in $W_5$ are not shown in (6). For example, trajectories of points of the block $\{011001\}$ can pass to the following 3-valent blocks: $\{001001\}$, $\{010001\}$, $\{011101\}$, and $\{011011\}$. The arrow $\{011001\} \to \{011000\}$ shows transition of trajectories between 5-valent blocks.

Let $F_0$, $F_1$, etc. be the intersections of adjacent blocks in the diagram (6):
$F_0 = \{111001\} \cap \{011001\}$, where $x_0 = 0$; $F_1 = \{011001\} \cap \{011000\}$, where $x_5 = 0$;
$F_2 = \{011000\} \cap \{011010\}$, where $x_4 = 0$; $F_3 = \{011010\} \cap \{011110\}$, where $x_3 = 0$;
$F_4 = \{011110\} \cap \{010110\}$, where $x_2 = 0$; ... $F_8 = \{100111\} \cap \{100101\}$, where $x_4 = 0$;
$\ldots F_{12} = F_0$.

Note, that on the face $F_0$ we have $x_0 = 0$, $x_1 > 0$, $x_2 > 0$, $x_3 < 0$, $x_4 < 0$, $x_5 > 0$.

We assume below that for all $j$

$$k_j = 1. \tag{7}$$

Consider as an example the block $\{011001\}$, where the equations of the system (3) have the form:

$$\dot{x}_0 = -x_0 - 1;\ \dot{x}_1 = -x_1 - 1;\ \dot{x}_2 = -x_2 - 1;$$
$$\dot{x}_3 = -x_3 - 1 + a_3;\ \dot{x}_4 = -x_4 - 1 + a_4;\ \dot{x}_5 = -x_5 - 1. \tag{8}$$

Trajectory of a point $P^{(0)} = (0, x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, x_4^{(0)}, x_5^{(0)}) \in F_0$ is described in the block $\{011001\}$ by the equations

$$x_5 = -1 + (1 + x_5^{(0)})e^{-t};\ x_0 = -1 + e^{-t};\ x_1 = -1 + (1 + x_1^{(0)})e^{-t};\ x_2 = -1 + (1 + x_2^{(0)})e^{-t};$$
$$x_3 = a_3 - 1 + (x_3^{(0)} + 1 - a_3)e^{-t};\ x_4 = a_4 - 1 + (x_4^{(0)} + 1 - a_4)e^{-t}; \tag{9}$$

or $x_j = c_j + \tau(x_j^{(0)} - c_j)$, where $\tau := e^{-t}$, and $c_j$ equals either $-1$, or $(a_j - 1)$. So, these equations describe a segment of a straight line.

If this trajectory starts at the point $P^{(0)} \in F_0$ and arrives at some moment $t = t_1$ at a point $P^{(1)} = (x_0^{(1)}, x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_4^{(1)}, 0) \in F_1$, then it follows from the equations (9) that coordinates of this point $P^{(1)}$ are represented in the following way:

$$x_0^{(1)} = -x_5^{(0)} \cdot e^{-t_1};\quad x_1^{(1)} = (x_1^{(0)} - x_5^{(0)}) \cdot e^{-t_1};\quad x_2^{(1)} = (x_2^{(0)} - x_5^{(0)}) \cdot e^{-t_1};$$

$$x_3^{(1)} = (x_3^{(0)} + (a_3 - 1)x_5^{(0)}) \cdot e^{-t_1}; \quad x_4^{(1)} = (x_1^{(0)} + (a_4 - 1)x_5^{(0)}) \cdot e^{-t_1}; \quad x_5^{(1)} = 0; \tag{10}$$

and $e^{-t_1} = (1 + x_5^{(0)})^{-1}$.

**Remark 2.** *In a similar way the system* (3) *and its solutions are represented in all other blocks of the partition* (4) *of the invariant domain* $Q$.

We study in this paper only those trajectories of the system (3) which do not intersect coordinate planes $\{x_j = 0\} \cap \{x_k = 0\} \subset \mathbb{R}^6$ of codimension 2. Thus, in each of these block $B$ trajectories of its points are linear, and each of these trajectories intersects the boundary of $B$ at an interior point $P^*$ of some face $F \subset \partial B$ as $t$ grows. We consider this point $P^*$ as an initial data of the system (3) in the block $B'$ such that $F = B \cap B'$, and so trajectories of the points continue to interiors of adjacent blocks.

In the large, these trajectories are piecewise linear and their vertices are located on the coordinate planes $x_j = 0$.

**Remark 3.** *Let us assign to each point* $P \in W_5 \setminus \{O\}$ *the ray* $OP$ *and its intersection with the boundary* $\partial Q^6 \approx S^5$. *For any* $t > 0$, *trajectories of some of the points of each of these faces should remain in* $W_5$. *Otherwise, the shifts along trajectories define a homotopy* $\partial Q^6 \to \partial Q^6 \setminus (\partial Q^6 \cap W_5)$, *and this contradicts to* $\pi_5(S^5) \approx Z$.

## 3 Invariant surface in non-invariant domain $W_5$

Just for simplicity of exposition, from now on we consider the system (3) under additional assumptions:

$$a_0 = a_2 = a_4; \quad a_1 = a_3 = a_5. \tag{11}$$

Let $a := a_{2i}$, $b := a_{2i+1}$,

Now, following [11, 19] we look for a point $P^{(0)} \in F_0$ such that after 4 steps in the diagram (6) along its trajectory it shifts to a point
$P^{(4)} = (x_0^{(4)}, x_1^{(4)}, 0, x_3^{(4)}, x_4^{(4)}, x_5^{(4)}) \in F_4$ such that for some positive $\mu$ we have:
$x_3^{(4)} = \mu x_1^{(0)}$, $x_4^{(4)} = \mu x_2^{(0)}$, $x_5^{(4)} = \mu x_3^{(0)}$, $x_0^{(4)} = \mu x_4^{(0)}$, $x_1^{(4)} = \mu x_5^{(0)}$.

Due to symmetry conditions (11), after 8 steps in the diagram (6) the point $P^{(0)}$ shifts along its trajectrory to a point $P^{(8)} \in F_8$ with coordinates $(x_0^{(8)}, x_1^{(2)}, x_2^{(8)}, x_3^{(8)}, 0, x_5^{(8)})$ such that $x_j^{(8)} = \mu x_{j-2}^{(4)} = \mu^2 x_{j-4}^{(0)}$; all subscripts are considered $(\mathrm{mod}\, 6)$. So, after 12 steps according to arrows of the diagram (6) such point $P^{(0)}$ shifts along its trajectory to the point $P^{(12)} \in F_0$ with coordinates $x_0^{(12)} = 0$, $x_1^{(12)} = \mu^3 x_1^{(0)}$, $x_2^{(12)} = \mu^3 x_2^{(0)}$, $x_3^{(12)} = \mu^3 x_3^{(0)}$, $x_4^{(12)} = \mu^3 x_4^{(0)}$, $x_5^{(12)} = \mu^3 x_5^{(0)}$. If $\mu = 1$, then trajectory of the point $P^{(0)}$ is a cycle and that 12 steps transformation $F_0 \to F_{12}$ could be called the Poincaré map of this cycle. However, this is not our case, see below.

For each face $F_m$ in the diagram (6), let us construct 5-dimensional coordinate system $(O; X_1^{(m)}, X_2^{(m)}, X_3^{(m)}, X_4^{(m)}, X_5^{(m)})$ as follows: Since the face $F_m$ is contained in the coordinate hyperplane $x_h = 0$, where $h + m \equiv 0\,(\mathrm{mod}\, 6)$, we put $X_1^{(m)} = \pm x_{1-m}$, $X_2^{(m)} = \pm x_{2-m}$, $X_3^{(m)} = \pm x_{3-m}$, $X_4^{(m)} = \pm x_{4-m}$, $X_5^{(m)} = \pm x_{5-m}$. The signs are choosen so that each face $F_m$ is contained in the positive orthant of the coordinate system $(O; X_1^{(m)}, X_2^{(m)}, X_3^{(m)}, X_4^{(m)}, X_5^{(m)})$.

In our calculations below, we identify the points $P^{(m)} \in F_m$ with their column vectors $OP^{(m)}$, $m = 0, 1, \ldots, 12$.

The representations (10) imply that if a point $P^{(0)} = (X_1^{(0)}, X_2^{(0)}, X_3^{(0)}, X_4^{(0)}, X_5^{(0)})$ shifts along its trajectory in the block $\{011001\}$ to $P^{(1)} \in F_1$, then

$$P^{(1)} = \frac{M_0 P^{(0)}}{(1 + X_5^{(0)})}, \quad \text{where} \quad M_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & b-1 \\ 0 & 0 & 0 & 1 & a-1 \end{pmatrix}. \tag{12}$$

If the point $P^{(1)}$ shifts in the block $\{011000\}$ along its trajectory to the point $P^{(2)} \in F_2$ which shifts in $\{011010\}$ to the point $P^{(3)} \in F_3$, and later shifts to $P^{(4)} \in F_4$, then for each of these steps we have $P^{(m+1)} = (1 + X_5^{(m)})^{-1} M_m P^{(m)}$ where the matrices $M_m$ are determined by analogues of formulas (10), see Remark 2. Thus, for some positive $Z_{IV} = Z_{IV}(X_5^{(0)}, X_5^{(1)}, X_5^{(2)}, X_5^{(3)})$ we have the following representation of this 4-steps transformation: $P^{(4)} = (1 + Z_{IV})^{-1} M_{IV} P^{(0)}$, where

$$M_{IV} = \begin{pmatrix} 0 & b-1 & -1 & 0 & 0 \\ 0 & -1 & a(b-1)^{-1} & -1 & 0 \\ 0 & 1-b & 0 & b(a-1)^{-1} & -1 \\ 0 & 1-a & 0 & 0 & a \\ 1 & -1 & 0 & 0 & 0 \end{pmatrix}$$

is the product $M_3 \cdot M_2 \cdot M_1 \cdot M_0$.

All the shifts $\varphi_m : F_m \to F_{m+1}$ are determined by fractional-linear transformations, similar to (12), and trajectories of most points of $F_m$ do not intersect $F_{m+1}$, since the diagram (6) is composed by 5-valent blocks. So, each ray $\rho_m = OP^{(m)} \subset F_m$ described in Remark 3 is transformed by the shift $\varphi_m$ to a ray $\rho_{m+1} = OP^{(m+1)} \subset F_{m+1}$. At the same time, the inverse matrix $M_{IV}^{-1}$ is "almost positive":

$$M_{IV}^{-1} = \begin{pmatrix} a\alpha\beta & b\alpha & a & 1 & 1 \\ a\alpha\beta & b\alpha & a & 1 & 0 \\ ab + b\alpha - 1 & b(b-1)\alpha & a(b-1) & b-1 & 0 \\ a\beta(a - b^{-1}) & ab-1 & a(a-1) & a-1 & 0 \\ a\beta & b & a-1 & 1 & 0 \end{pmatrix}.$$

Here $\alpha := a \cdot (a-1)^{-1}$, $\beta := b \cdot (b-1)^{-1}$.

**Lemma 3.1.** *The characteristic polynomial of $M_{IV}$*

$$\Pi(\lambda) = -\lambda^5 - \lambda^4 - \lambda^3 + \lambda^2 + \lambda \cdot (1 - ab\alpha\beta) + 1; \tag{13}$$

*has only one real root $\lambda_1$, and $\lambda_1 \in (0,1)$.*

The proof follows from $\Pi(0) = 1$, $\Pi(1) < 0$ and some calculations.

Let $e_1 \in F_0$ be the corresponding eigenvector of $M_{IV}$. Then the composition of the shifts $\Phi := \varphi_{11} \circ \varphi_{10} \circ \ldots \varphi_1 \circ \varphi_0 : F_0 \to F_0$ maps $e_1$ to $\lambda_1^3 (1+Z)^{-3} e_1$ where $Z > 0$. Since $\lambda_1^3 (1+Z)^{-3} < 1$, the map $\Phi$ transforms the ray $\rho_0 = OP^{(0)} \,||\, e_1$ to itself so that for any point $\bar{P}^{(0)} \in \rho_0$ we have $\Phi(\bar{P}^{(0)}) \in \rho_0$, and $|\bar{P}^{(0)}| > |\Phi(\bar{P}^{(0)})|$.

Thus, during 12 steps along the diagram (6), the points of this ray $\rho_0$ compose an invariant 2-dimensional piecewise linear surface $\Sigma \subset W_5$, and trajectories of all points of $\Sigma$ spiral towards the origin $O \in \Sigma$.

## Conclusions and future plans

So, we have proved the following

**Theorem 3.2.** *If $a > 1$, $b > 1$, then the system* (3), (7), (11) *has an invartiant surface $\Sigma \subset W_5$. Trajectories of points of this surface are attracted by the origin.*

**1.** Most of the agruments in the proof of this theorem remain valid without symmetry assumptions (11). In this case the equations of the system (3) and its solutions in the blocks of the partition (4) differ from (8) and (9) just by indices, thus the matrices $M_0$, $M_1$, $M_2$, $M_3$, $M_{IV}$, $M_{IV}^{-1}$, $M_{XII}$ etc. are not too bulky. Here XII corresponds to 12 steps in the diagram (6). One can verify that the matrix $M_{XII}^{-1}$ which describes transformation $\Phi^{-1}$ is strictly positive, as above, so the Frobenius-Perron theorem implies existence of an invariant surface $\Sigma \subset W_5$ in this asymmetrisc case as well. However, the characteristic polynomial in this case is much more complicated than $\Pi(\lambda)$ in (13).

**2.** The statement of the theorem seems to be true for the system (3) without assumptions (7) as well, cf. [7, 19]. In this case trajectories of the system are no longer piecewise linear, the transformations $\varphi_m : F_m \to F_{m+1}$ are not projective, but for any $t > 0$ trajectories of some of the points of each of these faces should remain in $W_5$, see Remark 3.

**3.** Similar construction can be reproduced for some circular chain of 3-valent blocks in $W_3$, analogous to the diagrams (5) and (6).

But since the partition of $Q^6$ contains 40 blocks of valence 3, the combinatorial structure of non-invariant domain $W_3$ is much more cumbersome than that of $W_1$ and $W_5$, so analysis of shifts along trajectories in these 3-valent blocks is not so transparent as for $W_1$ and $W_5$.

This work was supported by the Comprehensive Program for Fundamental Research of the Siberian Branch RAS II.1, integration project N 0324-2018-0011, and grant of RFBR 18-01-00057.

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